Matrices With Prescribed Lower Triangular Part

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ABSTRACT

We give a necessary and sufficient condition for the existence of a square matrix with prescribed characteristic polynomial and prescribed entries on and below the main diagonal.

Let F be a field and $A = [a_{i,j}] \in F^{n \times n}$. Let $f(x)$ be a monic polynomial over F of degree n. We are interested in the problem of obtaining conditions for the existence of a matrix $B = [b_{i,j}] \in F^{n \times n}$, with characteristic polynomial f, such that

$$b_{i,j} = a_{i,j} \qquad \text{for} \quad 1 \leqslant j \leqslant i \leqslant n. \tag{1}$$

In [2], we solved a similar problem where we only prescribe the entries (i, j) with $i > j$.

In this paper, we say that a matrix $C \in F^{s \times s}$ is *irreducible* if there exists no number $r \in \{1, \dots, s-1\}$ such that

$$C = \begin{bmatrix} C_{1,1} & C_{1,2} \\ 0 & C_{2,2} \end{bmatrix} \quad \text{and} \quad C_{1,1} \in F^{r \times r}.$$

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Consider A partitioned as follows:

$$A = \begin{bmatrix} A_{1,1} & & * \\ & \ddots & \\ 0 & & A_{k,k} \end{bmatrix}, \tag{2}$$

where $A_{t,t}$ is an irreducible matrix of size $s_t \times s_t$, $t \in \{1, \dots, k\}$.

The following theorem is our main result.

THEOREM. *The following conditions are equivalent:*

(a) *There exists a matrix $B = [b_{i,j}] \in F^{n \times n}$, with characteristic polynomial f, that satisfies* (1).

(b) *There exists a nonderogatory matrix $B = [b_{i,j}] \in F^{n \times n}$, with characteristic polynomial f, that satisfies* (1).

(c) *For each $t \in \{1, \dots, k\}$, there exists a polynomial*

$$f_t(x) = x^{s_t} + c_{s_t - 1}^{(t)} x^{s_t - 1} + \cdots \in F[x]$$

such that

$$\operatorname{tr} A_{t,t} = -c_{s_t - 1}^{(t)}$$

and

$$f = f_1 \cdots f_k.$$

We split the proof of this theorem into several lemmas. Given $C \in F^{n \times n}$, $i, j \in \{1, \dots, n\}$, we denote by $C(i|j)$ the submatrix that results from C on deleting the ith row and the jth column.

LEMMA 1. *Let $C = [c_{i,j}] \in F^{n \times n}$, and suppose that*

$$c_{i,i-1} = 1 \qquad \textit{for} \quad 2 \leqslant i \leqslant n,$$

$$c_{i,j} = 0 \qquad \textit{for} \quad 1 \leqslant j < i - 1 \leqslant n.$$

Then there exists an upper triangular matrix $T \in F^{n\times n}$, with the entries on the main diagonal equal to 1, *such that TCT^{-1} is a companion matrix of the form*

$$\left[\begin{array}{ccc} & * & \\ \hline 1 & 0 & 0 \\ & \ddots \quad \ddots & \\ 0 & 1 & 0 \end{array}\right]. \tag{3}$$

Proof. The proof is by induction on n. If $n = 1$, this result is trivial. Suppose that $n \geqslant 2$. Let $C_0 = C(1|1)$. According to the induction assumption, there exists an upper triangular matrix T_0, with the entries on the main diagonal equal to 1, such that $T_0 C_0 T_0^{-1}$ has the form (3). Let $[d_1 \ \cdots \ d_{n-1}]$ be the first row of $T_0 C_0 T_0^{-1}$. Let

$$T = \begin{bmatrix} 1 & d_1 & \cdots & d_{n-1} \\ 0 & & I_{n-1} & \end{bmatrix}([1] \oplus T_0).$$

Then TCT^{-1} has the form (3). ∎

Given $D \in F^{n\times n}$, $E \in F^{n\times p}$, we say that (D, E) is a *full-range* pair if the polynomial matrix

$$[xI_n - D| - E]$$

has all its invariant factors equal to 1. For general results concerned with this concept, see [1] and [3], for example. In [1], these results are stated for $F = \mathbb{C}$, the field of the complex numbers, although many of them can be applied to any other field. In [3], a full-range pair is said to be "completely controllable."

LEMMA 2. *Let $d_1 \in F$, $e \in F^{n\times 1} - \{0\}$. Then there exist $d_2, \ldots, d_n \in F$ such that*

$$\left(\begin{bmatrix} d_1 & d_2 & \cdots & d_n \\ 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}, e\right) \tag{4}$$

is a full-range pair.

Proof. Suppose that $e = [e_1 \ \cdots \ e_n]^t$. Let $m = \max\{i : e_i \neq 0\}$. The proof is by induction on m. If $m = 1$, then, for any $d_2, \dots, d_n \in F$, (4) is a full-range pair.

Now suppose that $m \geqslant 2$. Let

$$S = \left[\begin{array}{c|c} & -\dfrac{e_1}{e_m} \\ I_{m-1} & \vdots \\ & -\dfrac{e_{m-1}}{e_m} \\ \hline 0 & 1 \end{array}\right] \in F^{m\times m},$$

$$M = \left[\begin{array}{c|c|c} \begin{matrix} d_1 & 0 & \cdots & 0 \end{matrix} & & \\ \hline I_{n-1} & 0 & e \end{array}\right] \in F^{n\times(n+1)},$$

$$N = (S \oplus I_{n-m}) M \left(S^{-1} \oplus I_{n-m+1}\right).$$

Let C be the submatrix of N lying in rows and columns $1, \dots, m-1$. According to Lemma 1, there exists an upper triangular matrix $T \in F^{(m-1)\times(m-1)}$, with the entries on the main diagonal equal to 1, such that TCT^{-1} has the form (3). The (1, 1) entry of TCT^{-1} is equal to $d_1 - e_{m-1}/e_m$.

Recursively, we define matrices $R^{(h)} = [r_{i,j}^{(h)}] \in F^{n\times(n+1)}$, $h \in \{1, \dots, n - m + 1\}$, as follows:

$$R^{(1)} = (T \oplus I_{n-m+1}) N \left(T^{-1} \oplus I_{n-m+2}\right),$$

and, for $h \in \{1, \dots, n-m\}$,

$$R^{(h+1)} = U^{(h)} R^{(h)} \left(U^{(h)} \oplus [1]\right)^{-1},$$

where

$$U^{(h)} = \left[\begin{array}{c|c|c} & -r_{1,m+h-1}^{(h)} & \\ & \vdots & \\ I_{m+h-1} & -r_{m-1,m+h-1}^{(h)} & 0 \\ & 0 & \\ & \vdots & \\ & 0 & \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & I_{n-m-h} \end{array}\right].$$

According to the induction assumption, there exist $d_2', \dots, d_{m-1}' \in F$ such that (D_0, e_0), where

$$D_0 = \left[\begin{array}{cccc} d_1 - \dfrac{e_{m-1}}{e_m} & d_2' & \cdots & d_{m-1}' \\ \hline 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{array}\right], \qquad e_0 = \begin{bmatrix} 1 \\ r_{2,n}^{(n-m+1)} \\ \vdots \\ r_{m-1,n}^{(n-m+1)} \end{bmatrix},$$

is a full-range pair. Let

$$D' = \left[\begin{array}{cccc|cccc|c} & D_0 & & & & 0 & & & e_0 \\ \hline 0 & \cdots & 0 & 1 & r_{m,m}^{(n-m+1)} & r_{m,m+1}^{(n-m+1)} & & \cdots & r_{m,n}^{(n-m+1)} \\ \hline & 0 & & & & I_{n-m} & & & 0 \end{array}\right].$$

Let e' be the mth column of I_n multiplied by e_m. Clearly, (D', e') is a full-range pair. Let $D = V^{-1}D'V$, where

$$V = U^{(n-m)} \cdots U^{(2)}U^{(1)}(T \oplus I_{n-m+1})(S \oplus I_{n-m}).$$

We have $e = V^{-1}e'$. It is not hard to conclude that (D, e) is a full-range pair and has the form (4). ■

LEMMA 3. *Let $D_0 \in F^{(n-1)\times(n-1)}$, $b \in F^{(n-1)\times 1}$. If (D_0, b) is a full-range pair, then there exists a nonsingular matrix*

$$\begin{bmatrix} P & 0 \\ R & q \end{bmatrix} \in F^{n\times n}, \qquad \textit{with} \quad P \in F^{(n-1)\times(n-1)}, \tag{5}$$

such that

$$P^{-1}\begin{bmatrix} D_0 & b \end{bmatrix}\begin{bmatrix} P & 0 \\ R & q \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ I_{n-2} & 0 & 0 \end{bmatrix}. \tag{6}$$

Proof. Lemma 3 is a particular case of Theorem 2.11 of [3] or Theorem 6.2.5 of [1]. It can also be proved easily without using general results. ■

LEMMA 4. *Let* $D_0 \in F^{(n-1)\times(n-1)}$,

$$D = \begin{bmatrix} a & * \\ b & D_0 \end{bmatrix} \in F^{n\times n},$$

$e \in F^{n\times 1} - \{0\}$. *If* (D_0, b) *is a full-range pair, then there exists* $d \in F^{1\times(n-1)}$ *such that*

$$\left(\begin{bmatrix} a & d \\ b & D_0 \end{bmatrix}, e \right) \tag{7}$$

is a full-range pair.

Proof. Suppose that (D_0, b) is a full-range pair. According to Lemma 3, there exists a nonsingular matrix of the form (5) that satisfies (6). Let

$$e' = \begin{bmatrix} q & R \\ 0 & P \end{bmatrix}^{-1} e.$$

According to Lemma 2, there exists $d' \in F^{1\times(n-1)}$ such that (D', e'), where

$$D' = \left[\begin{array}{c|cccc} a - RP^{-1}b & & d' & & \\ \hline 1 & 0 & & & 0 \\ 0 & 1 & 0 & & \\ \vdots & & \ddots & \ddots & \\ 0 & 0 & & 1 & 0 \end{array}\right],$$

is a full-range pair. It is not hard to see that (D'', e), where

$$D'' = \begin{bmatrix} q & R \\ 0 & P \end{bmatrix} D' \begin{bmatrix} q & R \\ 0 & P \end{bmatrix}^{-1},$$

is a full-range pair and has the form (7). ■

LEMMA 5. *Let*

$$e = \begin{bmatrix} a_{2,1} & \cdots & a_{n,1} \end{bmatrix}^t.$$

If A is irreducible, then there exists $B_0 = [b_{i,j}] \in F^{(n-1)\times(n-1)}$ such that (B_0, e) is a full-range pair and

$$b_{i,j} = a_{i+1,j+1} \qquad \textit{for} \quad 1 \leqslant j \leqslant i \leqslant n-1. \tag{8}$$

Proof. The proof is by induction on n. If $n = 2$, this result is trivial. Suppose that $n \geqslant 3$. Let $A' = [a'_{i,j}]$ be the matrix that results from A adding the first column multiplied by a scalar α to the second column. The scalar α is chosen so that the matrix $C = [c_{i,j}] = A'(1|1)$ is irreducible. Let

$$b = \begin{bmatrix} a'_{3,2} & \cdots & a'_{n,2} \end{bmatrix}^t.$$

According to the induction assumption, there exists $D_0 = [d_{i,j}] \in F^{(n-2)\times(n-2)}$ such that (D_0, b) is a full-range pair and

$$d_{i,j} = c_{i+1,j+1} \qquad \text{for} \quad 1 \leqslant j \leqslant i \leqslant n-2.$$

According to Lemma 4, there exists $d \in F^{1\times(n-2)}$ such that (D, e), where

$$D = \begin{bmatrix} c_{1,1} & d \\ b & D_0 \end{bmatrix},$$

is a full-range pair. Then (B_0, e), where $B_0 = [b_{i,j}]$ results from D subtracting αe from the first column, is also a full-range pair. Clearly, the condition (8) is satisfied. ∎

LEMMA 6. *Let $B_0 \in F^{(n-1)\times(n-1)}$, $e \in F^{(n-1)\times 1}$. If (B_0, e) is a full-range pair, then there exist $d \in F^{1\times(n-1)}$, $a \in F$ such that*

$$\begin{bmatrix} a & d \\ e & B_0 \end{bmatrix}$$

is nonderogatory and has characteristic polynomial $f(x)$.

Proof. This lemma is a particular case of Corollary III, p. 143, of [3]. It can also be proved easily without using general results. ∎

LEMMA 7. *If A is irreducible and the coefficient of x^{n-1} in f is equal to $-\operatorname{tr} A$, then* (b) *is satisfied.*

Proof. It is a consequence of Lemmas 5 and 6. ∎

Proof of the theorem. Clearly (b) $\Rightarrow$ (a) $\Rightarrow$ (c). Now we suppose that (c) is satisfied in order to prove (b).

Let $t \in \{1, \dots, k\}$. Since $A_{t,t}$ is irreducible, there exists, according to Lemma 7, a nonderogatory matrix $B^{(t)} = [b_{i,j}^{(t)}] \in F^{s_t \times s_t}$ with characteristic polynomial f_t such that

$$b_{i,j}^{(t)} = a_{s_1 + \cdots + s_{t-1} + i,\, s_1 + \cdots + s_{t-1} + j} \qquad \text{for} \quad 1 \leqslant j \leqslant i \leqslant s_t .$$

Let $X_t \in F^{s_t \times s_t}$ be a nonsingular matrix such that

$$C_t = X^{(t)} B^{(t)} \left(X^{(t)} \right)^{-1}$$

is the companion matrix of f_t of the form (3). Let L_t be the $s_t \times s_{t+1}$ matrix that has its $(1, s_{t+1})$ entry equal to 1 and all the other entries equal to zero. Then the matrix

$$B = \left[b_{i,j} \right] = X^{-1} \begin{bmatrix} C_1 & & L_1 & & 0 \\ & \ddots & & \ddots & \\ & & C_{k-1} & & L_{k-1} \\ 0 & & & & C_k \end{bmatrix} X,$$

where

$$X = X^{(1)} \oplus \cdots \oplus X^{(k)},$$

is nonderogatory, has characteristic polynomial f, and satisfies (1). ∎

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